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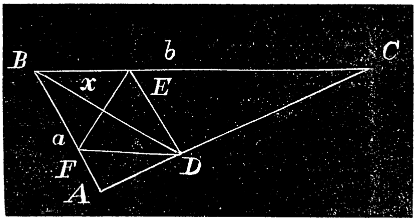


Draw BD bisecting the angle B; also draw DE and DF, respectively parallel to AB and BC. Then will $BEDF$ be an equilateral quadrilateral of which BD and EF are diagonals.

Let x represent the side of this quadrilateral. Then by similar trian.

$$(b-x):x=b:a;\ \therefore\ ab-ax=bx,\ \text{or}\ ax+bx=ab;$$

$$\therefore x=\frac{ab}{a+b},$$

which is the equation of the 4th proport'l existing between the lines a and b.

As the sides of a quadrilateral equilateral can be ascertained from the sides of the included angle, the diagonals of which are the measures of the angle, not by the degrees of an imaginary arc but by actual latitude and departure, two sides of a triangle with any data from which one of these diagonals may be determined, the unknown side and all other things about the triangle may be determined. That is, it solves all the cases of plane Trigonometry by measurement only, without the use of a table.

This method suggests a cheap and simple instrument for the field that would do all the work of the transit more simply and with equal accuracy. And for the observatory, where the quadrangle might be of any desir'd size, the latitude and departure to the $\frac{1}{100}$ part of every second in the semicircle might be read from the diagonals without the necessity of a single calcul'n.

SOLUTION OF PROB'S IN NO. 2; CONTINUED FROM P. 103.

306. "If N is an odd number and m the number of odd numbers less than N and prime to it (including 1), then N is a divisor of one of the two numbers $2^m\pm1$."

SOLUTION BY PROF. WM. WOOLSEY JOHNSON.

The number of odd numbers less than and prime to an odd number is one half of the whole number of numbers less than and prime to the given number; for, corresponding to every odd number r less than and prime to N, there is an even number $N-r$, also less than and prime to N. Hence $\varphi(N)$ denoting this whole number, the theorem is that when N is odd either

$$2^{\frac{1}{2}\phi(N)}-1 \qquad \text{or} \qquad 2^{\frac{1}{2}\phi(N)}+1$$

is divisible by N. But the theorem may be generalized thus, *If a is prime to N, one of the quantities*

$$a^{\frac{1}{2}\phi(N)}-1 \qquad \text{or} \qquad a^{\frac{1}{2}\phi(N)}+1$$

is divisible by N [except when $N=2$, the only case in which $\varphi(N)$ is an odd number].

By Euler's extension of Fermat's Theorem (see Serret's Course d'Algebre Supérieure, 3rd Ed., Vol. 2, p. 36)

$$a^{\phi(N)}-1 \text{ is divisible by } N, \tag{1}$$

and if $N=\alpha^n\beta^p\gamma^q \ldots$, $\alpha, \beta, \gamma \ldots$ being the *different* prime factors of N, we have $\varphi(N)=\alpha^{n-1}\beta^{p-1}\gamma^{q-1}\ldots(\alpha-1)(\beta-1)(\gamma-1)$,

whence if $N=N_1N_2$, where N_1 and N_2 are prime to one another, we have

$$\varphi(N)=\varphi(N_1)\,.\,\varphi(N_2). \tag{2}$$

Now unless N is a power of a prime or double the power of a prime, it will be possible to take N_1 and N_2 each greater than 2, and therefore $\varphi(N_1)$ and $\varphi(N_2)$ will both be even numbers. It follows that $\frac{1}{2}\varphi(N)$ is a common multiple of $\varphi(N_1)$ and $\varphi(N_2)$ and therefore $a^{\frac{1}{2}\phi(N)}-1$ is divisible by $a^{\phi(N_1)}-1$ and $a^{\phi(N_2)}-1$; hence by (1) it is divisible by N_1 and by N_2, and since N_1 and N_2 are prime to one another, it is divisible by N_1N_2, that is by N.

The cases excluded above are those in which N is of the form α^n, $2\alpha^n$ or 2^r, α denoting an odd prime, and $r>1$.

We have $$a^{\phi(N)}-1=[a^{\frac{1}{2}\phi(N)}-1][a^{\frac{1}{2}\phi(N)}+1]. \tag{3}$$

Now if $N=\alpha^n$, since by (2) the first member of (3) is divisible by α^n, one of the factors in the second member is divisible by the prime α, and since these factors differ by 2 units, the other is *not* so divisible; therefore the former factor is divisible by α^n that is by N.

Again if $N=2\alpha^n$ the same proof shows that one of the factors is divisible by α^n, but since in this case the first member of (3) is divisible by $2\alpha^n$, one of the factors is divisible by 2, and therefore since they differ by 2 *both* factors are so divisible, and hence one of them is divisible by $2\alpha^n$ i. e. by **N**.

Again if $N=4$, both the factors in the second member of (3) are proved as above to be even and since they differ by 2 one of them is divisible by 4.

Finally if $N=2^r$ where $r>2$, a, which is by hypothesis prime to N, is odd, and we may write $a=2^2k\pm1$, where k is an integer. Squaring successively we have

$$\begin{aligned} a^2 &= 2^3k_1+1,\\ a^{2^2} &= 2^4k_2+1,\\ &\cdots\cdots\\ a^{2^{r-2}} &= 2^rk_{r-2}+1. \end{aligned}$$

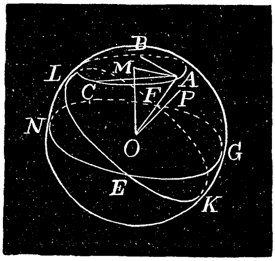
Now in this case $\varphi(N)=2^{r-1}$ and $\frac{1}{2}\varphi(N)=2^{r-2}$, hence the final equation expresses that

$$a^{\frac{1}{2}\phi(N)}-1=2^rk_{r-2}=Nk_{r-2}.$$

307. "Three points are taken at random in the surface of the northern hemisphere; find the chance that the small circle through the points lies wholly north of the equator, supposing the earth a perfect sphere."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let *NEGF* be the equator, *A*, *B*, *C*, three random points in the surface of the northern hemisphere, *M* the center of the small circle through *A*, *B*, *C*, *O* the center of the earth, and *OP* a radius such that *AB* is parallel to the plane *MOP*. From *P* as a pole, describe *EKFL*, the circumference of a great circle.



Now instead of confining the three points to the surface of the hemisphere whose great circle is *NEGF*, we will confine them to the surface of the hemisphere whose great circle is *EKFL*, and consider the number of ways they can be taken, so that the circle through them shall lie wholly in the hemisphere. This will evidently give the same result as that required in the question.

Let $OA = r$, $\angle AOM = \theta$, $\angle BAM = \varphi$, $\angle CAM = \psi$, $\angle MOP = \mu$, and the angle the plane *MOP* makes with a fixed plane through $OP = \omega$. Then an element of surface at *A* is $2\pi r^2 \sin\theta d\theta$, at *B* it is $4r^2 \sin\theta \cos\varphi d\varphi d\mu$, and at *C* it is $4r^2 \sin\theta \sin\mu \cos\psi \sin(\varphi+\psi) d\psi d\omega$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$, and doubled; of μ, 0 and $\frac{1}{2}\pi - \theta$; of ω, 0 and 2π. Hence, since the whole number of ways the three p'ts can be taken in the surface of a hemisphere is $(2\pi r^2)^3$, the required chance is,

$$\frac{2}{(2\pi r^2)^3}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\int_0^{2\pi} 32\pi r^6 \sin^3\theta d\theta \cos\varphi d\varphi \cos\psi \sin(\varphi+\psi) d\psi \times \sin\mu\, d\mu d\omega$$

$$= \frac{16}{\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta} \sin^3\theta d\theta \cos\varphi d\varphi \cos\psi \sin(\varphi+\psi) d\psi \sin\mu d\mu$$

$$= \frac{16}{\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\varphi}^{\frac{1}{2}\pi} (1-\sin\theta)\sin^3\theta d\theta \cos\varphi \cos\psi \sin(\varphi+\psi) d\psi$$

$$= \frac{8}{\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [(\tfrac{1}{2}\pi+\varphi)\sin\varphi + \cos\varphi](1-\sin\theta)\sin^3\theta d\theta \cos\varphi d\varphi$$

$$= 6\int_0^{\frac{1}{2}\pi} (1-\sin\theta)\sin^3\theta d\theta = 4 - \tfrac{9}{8}\pi.$$

[Mr. Adcock obtains, for the required probability, $2\pi(\frac{2}{3} - \frac{3}{16}\pi)$, the numerical value of which does not differ much from that obtained by Prof. S. The method he employs, however, is similar to that of his published solution of 286, which is incorrect, as we have before stated, because the units in the numerator and denominator are dissimilar, and therefore not necessarily equally probable. For instance, a finite number of units in the numerator contains, as individual units, many *concentric* circles; whereas it may easily be shown that the chance that there will be *two* concentric circles in a finite number of units of the denominator, is infinitely small.

Mr. Adcock requests us to make the following corr. in his solution of 286: "For number of positions use the number of combinations instead of the number of permutations. For instance, for $\pi^3m^3r^6$ read, $\frac{1}{6}\pi^3m^3r^6$."—Ed.]

308. "Evaluate

$$\int_0^{\frac{a}{2}} dx \sin[x(a-x)] \text{ and } \int_0^{\frac{a}{2}} dx \cos[x(a-x)]$$

by a rapidly converging series."

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Since $\sin(ax-x^2) = ax-x^2-\frac{(ax-x^2)^3}{1.2.3}+\frac{(ax-x^2)^5}{1.2.3.4.5}-$&c., and by p'ts,

$$\int(a-x)^nx^ndx = -\frac{(a-x)^{n+1}x^n}{n+1}-\frac{n(a-x)^{n+2}x^{n-1}}{(n+1)(n+2)}-\frac{n(n-1)(a-x)^{n+3}x^{n-2}}{(n+1)(n+2)(n+3)} -\text{\&c.}+C.$$

$$\therefore \int_0^{\frac{1}{2}a}(a-x)xdx = -\tfrac{1}{2}(a-x)^2x-\tfrac{1}{6}(a-x)^3+C = \tfrac{1}{12}a^3, \text{ and}$$

$$\int_0^{\frac{1}{2}a}(a-x)^3x^3dx = \frac{a^7}{280}, \int_0^{\frac{1}{2}a}(a-x)^5x^5dx = \frac{a^{11}}{5324}; \text{ hence}$$

$$\int_0^{\frac{1}{2}a} dx\sin[x(a-x)] = \frac{a}{12}-\frac{a^7}{1680}+\frac{a^{11}}{106480}-\text{\&c.},$$

which converges rapidly for values of a not greater than one.

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

Solutions of problems in No. 3 have been received as follows:

From Geo. H. Bangs, 311; Alex. S. Christie, 310; Prof. W. P. Casey, 309, 310, 311; Geo. Eastwood, 310; Prof. E. J. Edmunds, 309, 310; H. Heaton, 310, 311, 312; Lieut. Jas. M. Ingalls, 312; Prof. J. H. Kershner, 309, 310, 311, 312 [Prof. Kershner should have been credited in No. 3 for a solution of 304, which was omitted by an oversight]; W. L. Marcy, 311; Prof. D. J. Mc Adam, 311; P. Richardson, 311; Prof. J. Scheffer, 310, 311, 312; J. L. Schock, 312; Prof. De Volson Wood, 311.

309. "In a given circle, find the vertices of the inscribed square, pentagon, octagon, and decagon by using the dividers alone."